MATH-F410 – REP. DES GROUPES & APP. A LA PHYS.

Solutions des exercices 2022-2023

Séance 5 : Représentations des groupes cristallographiques et groupes de Coxeter

1. $R_{\{3,3\}}$ and the even permutations A_4

Section a)

The following figure displays two important axes of symmetry to understand the structure of $R_{\{3,3\}}$:

Although maybe not completely clear from the picture, the red axis goes through the middle of face 1 3 4 up to vertex 2. It is clearly an order 3 axis (it rotates the triangle 1 3 4 by $2\pi/3$). There are 4 of these axes, one going through each vertex. On the other hand, the blue axis connects the centers of edges 1 4 and 2 3. This is an axis of order 2 (rotation by π) which sends $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$. There are three of these, one for each pair of edges. Clearly these transformations on the tetrahedron can be mapped to permutations within S_4 just by looking at the labels of the vertices:

- For the red axis, rotation by $2\pi/3$ is mapped to $(1\,4\,3)$; while rotation by $4\pi/3$ is mapped to $(1\,3\,4)$ (these are inverse operations).
- For the blue axis, rotation by π is mapped to $(1\,4)(2\,3)$.

This identification gives a bijective map from $R_{\{3,3\}}$ to A_4, the set of *even* permutations within S_4. Recall that even permutations are formed by an even number of transpositions, as is the case for $(1\,4)(2\,3)$ and $(1\,4\,3) = (1\,3)(1\,4)$, for example. Even permutations within S_4 are 3-cycles, products of two disjoint transpositions, and the identity element. All these are obtained as elements of $R_{\{3,3\}}$ through the previous identification. This clearly gives an isomorphism, since the product structure of permutations is equivalent to what we do when we apply symmetry transformations to the tetrahedron and read how the vertices are reshuffled. We conclude $R_{\{3,3\}} \cong A_4$, and from now on we will work with A_4.

Section b)

Notice that conjugacy classes within A_4 are not the same as conjugacy classes within S_4 (we are now only allowed to conjugate by A_4 elements). In particular, we cannot use the result for the permutation groups S_n that conjugacy classes are given by the structure of disjoint cycles of the elements. You can try to understand this with a simple example which is useful for this question. Take $(1\,2\,3) \in A_4$. In S_4, conjugating by a transposition $(1\,2)$, we get $(1\,2)(1\,2\,3)(1\,2)^{-1} = (1\,3\,2)$. So $(1\,2\,3)$ and $(1\,3\,2)$ are conjugate in S_4 (this we already knew). The relevant point is that it is impossible that they are conjugate in A_4. Just try to find an element of A_4 that acting by conjugation on $(1\,2\,3)$ gives $(1\,3\,2)$: you will not find it. To solve the question, there are two possible approaches. First, in the spirit of the example we just discussed, we can try to conjugate by all elements of A_4 and identify the conjugacy classes. Proceeding this way, we find that there are four:

$$
\begin{aligned}
&[e] = \{e\} , && [(1\,2)(3\,4)] = \{(1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\} , \\
&[(1\,2\,3)] = \{(1\,2\,3), (1\,4\,2), (1\,3\,4), (2\,4\,3)\} , && [(1\,3\,2)] = \{(1\,3\,2), (1\,2\,4), (1\,4\,3), (2\,3\,4)\} .
\end{aligned}
$$

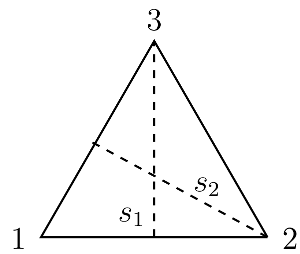
It is reasonable to think that there must be a more direct way to find this result, without going through the whole process of conjugating by all possible different elements in the alternating group. Indeed there is, but it requires proving an auxiliary result: a conjugacy class in S_n splits into two distinct, equal size ones in A_n if and only if its elements are formed by disjoint cycles of distinct, odd length. You can find the proof here. This explains why the conjugacy class of, e.g., $(1\,2\,3)$ splits in A_4 but that of $(1\,2)(3\,4)$ does not. There is also a physically intuitive way of looking at this, specially for this group which is relatively simple. Conjugation is like a "change of basis" using one of the elements of the group. If we are looking at the alternating group, we are considering elements which are part of $SO(3)$, therefore rotations (we do not allow reflections). Clearly, the rotations $(1\,2\,3)$ and $(1\,3\,4)$ are conjugate within $A_4 \cong R_{\{3,3\}}$: just rotate the figure in the previous page around the vertical axis (meaning conjugate by $(2\,3\,4) \in A_4$). On the other hand, it is also clear that $(1\,2\,3)$ and $(1\,3\,2)$ cannot be related by conjugation, because the geometric operation that relates them is a reflection (corresponding to, e.g., conjugation by $(1\,2)$).

Section c)

A direct application of Burnside's theorem gives, using $|A_4| = 12$, $12 = 1 + n_1^2 + n_2^2 + n_3^2$. The only possible solution is $n_1 = n_2 = 1$, $n_3 = 3$. Thus, we have three one-dimensional irreps (the trivial one and two extra ones) and a three-dimensional representation (that which views A_3 as rotations in space).

2. D_3 generated by reflections

In the figure at the top of next page we identify two reflections, s_1 and s_2, which are symmetry transformations of the triangle. Furthermore, let us call r the elementary rotation by $2\pi/3$ in counterclockwise direction, and r^{-1} the inverse (i.e., $2\pi/3$ rotation in clockwise direction or equivalently $4\pi/3$ in counterclockwise direction). There is a nice identification also of these elements as permutations of S_3: looking at the figure, $s_1 = (1\,2)$, $s_2 = (1\,3)$, and $r = (1\,2\,3)$. Now, either by looking at the geometric action of



the different transformations, or by simply multiplying permutations, it is easy to show that:

- $s_1 s_2 = r^{-1}$ and $s_2 s_1 = r$.
- $s_1 r = s_1 s_2 s_1 = s_3$ (s_3 is the remaining reflection not drawn in the figure).

This already shows that the full group D_3 can be written in terms of the basic reflections s_1 and s_2. Furthermore, the final part of the question is shown by computing $(s_1 s_2)^3 = (r^{-1})^3 = e$.

3. Coxeter groups

Section a)

This is immediate from the definition with $m_{ij} = 2$: $e = (s_i s_j)^2 = s_i s_j s_i s_j$. Since the s_k are reflections ($s_k^2 = e$), we get $s_i s_j = s_j s_i$ and the elements commute if $m_{ij} = 2$.

Section b)

This is a generalization of the previous exercise. Consider a regular polygon with p vertices, and put it in a way which is symmetric with respect to reflections along the x-axis (this is just a convenient choice of axes). D_p is the symmetry group of this polygon. Let R be the fundamental rotation of D_p (i.e., rotation by $2\pi/p$ in counterclockwise direction which moves a vertex of the given regular polygon to the position of the next one). Let $F(\alpha)$ be a reflection along an axis which forms an angle α with the x axis. We know by construction that $F(0)$ is a symmetry, call it s_1 (clearly, $s_1^2 = e$). The following reflection symmetry is $s_2 = F(\pi/p) = R s_1$ (the only non-trivial part is to get convinced of this identity, notice that $R s_1$ as we have defined it leaves the line forming an angle π/p with the x-axis invariant, and it must be a reflection because it is a product of reflection and rotation, thus the result follows). It follows that $R = s_2 s_1$, therefore $(s_2 s_1)^p = e$ (and no power less than p gives the identity, because the fundamental rotation has order p). Now, the group is completely generated by s_1 and s_2: $R = s_2 s_1$ generates all rotations, while reflections are obtained by a generalization of the previous result,

$$F(\pi k/p) = RF(\pi(k-1)/p) = R^k F(0) = (s_2 s_1)^k s_1 \quad \forall k = 0, \dots, p-1 \ .$$

Thus D_p is just (isomorphic to) the Coxeter group generated by two reflections, s_1 and s_2, joined by a line of order p (meaning $(s_2 s_1)^p = e$).

Section c)

Take $s_1 = (1\,2), s_2 = (2\,3), \ldots, s_i = (i\; i+1), \ldots s_n = (n\; n+1)$. These are all reflections (in the sense $s_i^2 = e$), and furthermore they satisfy the relations of the Coxeter group with the structure given in the question:

- If $|j - i| > 1$, the product $s_i s_j = (i\; i+1)(j\; j+1)$ has order 2.
- $s_i s_{i+1} = (i\; i+1)(i+1\; i+2) = (i\; i+1\; i+2)$ has order 3.

Finally, the permutations $s_1, \ldots, s_n$ generate the full S_{n+1}. This can be seen because any transposition can be obtained by conjugation inside that set of reflections (e.g., $(1\,3) = (2\,3)(1\,2)(2\,3)$), and then using the result that any permutation can be written as a product of transpositions.

Section d)

This is a result presented in the lecture notes. For a crystallographic group, $s_i s_j \equiv r$ is a rotation (element of $SO(n)$) which preserves a certain lattice. If it is a Coxeter group, $(s_i s_j)^{m_{ij}} = e$, so r is a rotation of order m_{ij}. Consider the trace of r written as a matrix. This is basis-independent. We can compute it in an adapted orthonormal basis for which r acts on the plane defined by the first two vectors, thus obtaining $\mathrm{Tr}(r) = 2\cos\phi$, with ϕ the angle of the rotation implemented by r. Thus, $\phi = 2\pi/m_{ij}$, and $\mathrm{Tr}(r) = 2\cos(2\pi/m_{ij})$. On the other hand, we can also compute the trace in the basis adapted to the lattice, in which case the matrix has only integer entries (this is the condition that it must map the lattice to itself, any vector of the basis must be carried by r into a linear combination with integer coefficients of the basis lattice vectors). Thus $2\cos(2\pi/m_{ij}) \in \mathbb{Z}$. The only possibilities are

- $2\cos(2\pi/m_{ij}) = 2$, thus $m_{ij} = 1$. In terms of the Coxeter group this is meaningless, because we would be saying that $s_i s_j = e$, so actually $s_i = s_j$ (we were not considering two different reflections in the first place).
- $2\cos(2\pi/m_{ij}) = 1$, thus $m_{ij} = 6$.
- $2\cos(2\pi/m_{ij}) = 0$, thus $m_{ij} = 4$.
- $2\cos(2\pi/m_{ij}) = -1$, thus $m_{ij} = 3$.
- $2\cos(2\pi/m_{ij}) = -2$, thus $m_{ij} = 2$.